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CONCEPTUAL UNDERSTANDING OF FUNCTIONS: A TALE OF TWO SCHEMAS[1]

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Abstract: *Concepts-first* proponents propose that children are born with potential for conceptual knowledge in a domain and use this knowledge to generate and select procedures for solving problems in that domain. *Procedures-first* theorists propose that children first learn procedures for solving problems in a domain and then derive concepts from repeated experiences with solving those problems. A third view is that conceptual and procedural knowledge develop iteratively and influence each other. We propose a view that is based on Case's theory of intellectual development and that builds on and extends the third view. We suggest three ways in which conceptual and procedural knowledge relate to each other and support our view with empirical work on children's learning of mathematical functions.

Much has been written about the distinction between conceptual and procedural understanding in mathematics learning (e.g., Anderson, 1993; Greeno, et al., 1984; Hiebert, 1986). Considerable attention has also been given to whether conceptual or procedural knowledge emerges first (e.g., Gelman & Williams, 1998; Siegler, 1991; Sophian, 1997). *Concepts-first* proponents propose that children are born with potential for conceptual knowledge in a domain and use this knowledge to generate and select procedures for solving problems in that domain (e.g., Gelman & Meck, 1983; Halford, 1993). *Procedures-first* theorists propose that children first learn procedures for solving problems in a domain and then derive concepts from repeated experiences with solving those problems (e.g., Karmiloff-Smith, 1992; Siegler & Stern, 1998). A third view is that conceptual and procedural knowledge develop iteratively (e.g., Fuson, 1988; Rittle-Johnson et al., in press) and influence each other, with an increase in one type of knowledge leading to an increase in the other type, which stimulates an increase in the first, et cetera.

In this theoretical paper, we propose a view that builds on and extends the third view. Our view is based on Case's theory of intellectual development. It will be exemplified in this paper by empirical work on children's learning of mathematical functions. To emphasize knowledge in action, we will also use the terms "understanding" and "doing" for "conceptual knowledge" and "procedural knowledge," respectively. We suggest three ways in which conceptual knowledge (*understanding*) and procedural knowledge (*doing*) relate to each other: (1) *Understanding* and *doing* mathematics are always simultaneously present in any activity, but the proportion varies. (2)

Understanding and *doing* each arise initially within numeric/sequential and within spatial/analogic aspects of a domain, but then begin to relate across these aspects. (3) Developmentally, a mathematical activity that is viewed as a conceptual accomplishment, or *understanding*, at one age may be viewed primarily as *doing* at another age. The second and third ways of relating *understanding* and *doing* will be described after the theoretical framework is developed.

Relating *Understanding* and *Doing*

Most mathematical actions do not reflect only *understanding* or only *doing*. Rather, *understanding* and *doing* are always present in some proportion, with one or the other foregrounded. For example, people who understand why a particular algorithm works do not necessarily use those understandings while carrying out that algorithm, but they could shift into explaining mode and draw on these understandings. Similarly, it is not usually accurate to say that students who have learned an algorithm by rote understand nothing about it. Thus, it is more helpful always to consider that a mixture of *understanding* and *doing* are present or potentially present even when one is foregrounding *doing* or *understanding*.

The theoretical perspective we are taking for looking at how *understanding* and *doing* are related is that of Case's theory of intellectual development (e.g., Case, 1992, 1996). In Case's view, a deep conceptual understanding in any domain of learning is rooted in what he called a *central conceptual structure*. Central conceptual structures are central organizing features of children's domain-specific cognitive processing. They are constructed by integrating two primary mental schemas. The first of these is digital and sequential, and the second is spatial or analogic. In the first of four hypothesized phases of children's learning in a domain, each of these two primary schemas is elaborated in isolation. In the second phase they become integrated. The result is that a new psychological unit is constructed – the central conceptual structure – which constitutes deep conceptual understanding in that domain. During Phases 3 and 4, further learning and development (both in terms of biological maturation and learning experiences) build on students' understandings by elaborating further their numeric and spatial knowledge within the context of an already integrated understanding (a central conceptual structure).

Case's theory suggests the second and third aspect of the relationships between *understanding* and *doing*. As the second aspect, Case identified as conceptual understanding only those understandings that relate across numeric and spatial aspects of a domain. We also think that specifying conceptual understanding as related across numeric and spatial aspects is a useful point of view. However, we believe that a combination of *understandings* and *doings* are present in some proportion throughout each of these four stages of development. Therefore, we suggest the term "integrated understandings" or "integrated doings" for those *understandings* and *doings* that relate the numeric and spatial aspects.

For example, in Phase 1, each of the still-isolated primary numeric and spatial schemas includes both *understandings* and *doings*. In Phase 2, when the numeric and spatial schemas integrate to form a central conceptual structure the *understandings* and *doings* that were associated with each independent schema also integrate and become "integrated understandings" and "integrated doings". In Phase 3, variants of the Phase 2 central conceptual structure are developed as children apply their new structure to novel situations. These new Phase 3 structures involve further elaborations of either the embedded numeric or spatial structure. With these structural elaborations come sophistications in children's schema-relevant *understandings* and *doings* (i.e., spatial or numeric schema). Then, in Phase 4, numeric and spatial *understandings* and *doings* that are even further elaborated become the primary schemas that will integrate to form a central conceptual structure at the next level of development. That is, what is considered Phase 4 for one central conceptual structure is also Phase 1 for the next central conceptual structure.

The above model of development is best considered as an optimal learning sequence that should be supported and promoted through carefully designed instructional approaches. This view has already been proven effective in stimulating experimental learning units or curricula that result in powerful learning (e.g., Griffin & Case, 1996; Kalchman & Case, 1998, 1999; Moss & Case, 1999).

The third aspect of relationships between *understanding* and *doing* stems from Case's theory when one is looking across age levels of development. This aspect is that the same internal mental schema simultaneously indicates integrated conceptual *understandings* and *doings*, if viewed from one level, and not-yet-integrated conceptual *understandings* and *doings*, if viewed from the next level. This shifting across ages in viewing a mathematical accomplishment as *understanding* at one age but as *doing* at another age is fairly common in considering mathematical thinking. For example, learning that the last count word tells you "how many" (the count-cardinal principle) is a major conceptual accomplishment for 3 or 4 year-olds, but we take this understanding for granted for older children and consider their counting to be primarily *doing*, even though they automatically use this count-cardinal understanding.

Two Schemas-One Structure in the Domain of Functions

The topic of functions has been widely recognized as being central and foundational to mathematics in general. Literature indicates, however, that students of all ages have difficulty mastering the topic using traditional instruction approaches. The roles of numeric and spatial understanding in this domain are critical given that a concern among mathematics educators is that students have difficulty not only with moving among representations of a function (e.g., table, graph, equation, verbal description) (e.g., Goldenberg, 1995; Leinhardt et al., 1990; Markovitz et al., 1986), but also with understanding how and why the function concept is "representable" in tables, graphs, and equations (Thompson, 1994). Each of these representations embodies both spatial and numeric aspects of any function.

Two particularly relevant objectives have been among the major goals of our recent work. First, we have been working toward a comprehensive and coherent theoretical model for how students come to develop over time *understandings and doings* for functions, and how particular spatial and numeric characteristics of various functions influence development (Kalchman, 2001; Kalchman et al., 2001). Second, the first author has developed a curriculum intended to help students construct a central conceptual structure in a manner consistent with the proposed developmental sequence described in the model for learning (Kalchman, 2001; Kalchman & Case, 1998, 1999; Kalchman, Moss, & Case, 2001).

When students have a numeric understanding of functions, they can carry out calculations for making a table of values from an equation and plot the resultant coordinate points. This sort of understanding may be likened to what has been called a "process" or procedural understanding of functions (e.g., Kalchman & Case, 2000; Sfard, 1992). With this numeric understanding, students can use algorithms for finding, for example, the slope or y-intercept of a function. When students have a spatial understanding of functions, they can make qualitative judgements about the general shape of the graph of a function (e.g., straight or curved) or assess the magnitude of the slope of a function by comparing its steepness or direction (i.e., increasing or decreasing) to benchmark functions such as $y = x$ and $y = x^2$ (Confrey & Smith, 1994; Kalchman, 2001). This spatial understanding does not ensure that students have the computational skills for doing accurate quantitative comparisons.

When students have an integrated conceptual understanding, they can recognize and relate the spatial and numeric implications of the function concept in general and of each representation in particular. For example, a table of values is primarily numeric and sequential. However, there are spatial patterns that can be used for finding the slope, or even the overall shape, of a function: If the y-values in a table increase by 2 for every unit change in x, students can evaluate the pattern between successive y values, which is a constant increase of 2. From this pattern they can discern that the function must be linear with a relative steepness, or slope, of 2. This pattern can then be generalized into a symbolic expression of $y = 2x$.

Influences of Instruction on Students' Integrated Understanding of Functions

In a recent analysis of two modern secondary-level textbook units on functions (Kalchman, 2001), a strong emphasis was identified only on the development of numeric, sequential knowledge in the form of algorithms and mathematical notations. Spatial elements were barely addressed, and when they were, they were generally shown as isolated representations that *resulted* from numeric/algebraic procedures. That is, a graph of a function might be generated by carrying out calculations for finding coordinate pairs, but the graph itself was not an object of mathematical inquiry or thought. This sort of limited instruction cannot support students' developing of a wellconstructed, balanced, and integrated conceptual framework for the domain. Without such a framework, students will have difficulty in simultaneously *doing* and *understanding* more advanced mathematics such as calculus.

Students' opportunities for constructing such an integrated conceptual framework are greatly increased when they experience instruction and curricula that focus on developing both numeric and spatial *understandings* and *doings* and ensure ample opportunity for integrating all of these. One such curriculum was developed by Kalchman and Case and has been shown experimentally to help students construct deeper and more flexible understandings of functions than do students who learn from textbooks (Kalchman & Case, 1998, 1999). In this integrative curriculum, the context of a walkathon is used to bridge students' spatial and numeric understandings and to help them foster a central conceptual structure for the domain (see Kalchman, 2001, for a description of this curriculum).

To illustrate differences in students' reasoning about functions following a textbook unit to that of the "walkathon" approach, we will exemplify differences between an integrated conceptual understanding of select functions problems and an understanding that favors the numeric, sequential aspect of the domain. We will use examples of how students in an advanced-level Grade 11 mathematics class (n = 17), who had at least three years of textbook-based instructions in functions, responded to two tasks. These examples will be compared to responses to those same items by students in a high-achieving Grade 6 sample (n = 48), who had experienced three weeks of the "walkathon" curriculum.

Methods

Forty-eight Grade 6 students and 17 Grade 11 students were involved in the present study. All students attended the same independent school north of a major urban center in Canada. The Grade 6 students comprised two intact classes at the school. Each of these sixth-grade classes had 12 classes of experimental instruction at about 50 minutes each for a total of 600 minutes of instructional time. The first-author of this paper taught these students. The Grade 11 students were an intact class at the same school. This group had 15 classes of a standard textbook unit of functions, with each class being 80 minutes long for a total of 1200 minutes of instructional time. The regular classroom teacher taught these students. Each class of students was given the same set of problems to complete both before and after the respective instructional units. Only posttest responses will be presented and discussed here because pretest results were so low for both groups and the effects of instructional type are most interesting.

Tasks and Results

The tasks shown here are from a twelve-item functions test developed for a larger study (the complete test may be found in Kalchman, 2001). Both of the tasks below represent items designed to test for students' first integrated understanding of functions, i.e., their baseline integrated, conceptual understanding of functions.

In the first task (see Figure 1), students were asked to give an equation for a function that would cross through $y = x + 7$, which was shown graphically in the upper-right quadrant of a grid with a unitized scale from 0 - 10 on each axis. Students were told that their function had to pass through the given function within the observable space. We decided to have students work within the observable space in order to test their abilities to generate specific (albeit partial) representations of a more general function (Schwartz & Dreyfus, 1995). The solution space for this item is infinite. As a result of instructional experiences at both grade levels, students were likely to generate one of three particular types of functions: increasing functions with slopes steep enough to pass through the given function (with y-intercepts ranging from $0 \leq b \geq 7$); decreasing functions with y-intercepts ≥ 7; or increasing or decreasing curving functions again with a number of different y-intercepts possible.

Fifty-three percent of the older students and 73% of the younger students gave a correct solution for this item. A qualitative analysis of the approaches taken to the problem by each group overall also indicates how most of the younger students were

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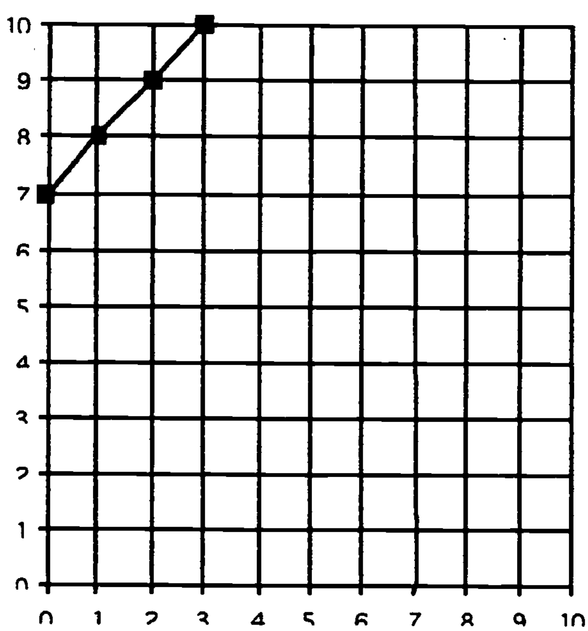
Can you think of a function that would cross the function seen in the graph below? What is the equation of the function you thought of?



Explain why the equation you chose is a good one.

Figure 1. Item 1.

using an integrated conceptual framework that involved an interactive dynamic of *understandings* and *doings* and how the older students were using primarily just a numeric approach.

A common error among the eleventh graders was first to use algorithms to find the slope and y-intercept of the graphed function and then to give the slope/intercept form of the equation. Students then substituted the negative reciprocal of the slope into the equation to get . Although this equation may be considered a correct solution, many students then proceeded to draw a line perpendicular to the one given to represent this new function, but with y-intercepts greater than 7. These students explained that "the line is perpendicular so it works." At first glance the approach seems sophisticated and suggests conceptual understanding. However, many of these students relied heavily on algorithms and did not recognize, or at least acknowledge, the algebraic implications of moving the y-intercept on the graph.

The younger students seemed to approach the problem from a more conceptually integrated point of view. Many of these students first drew a line that passed through the original one, and then derived the equation from the information on the graph. For example, one student drew a line from (0, 10) to (5, 0), made a table of values, wrote the equation $y = x$*-2 +10 and explained "Because it goes down by –2 [sic], so if it starts at 10 it will pass through the line." This sort of solution suggests an integrated conceptual approach to the task in that she is using a numeric approach for deriving the equation from the number pattern found on the graph while connecting it to the spatial entailments of the problem. Operating numerically and spatially suggests simultaneous procedural and conceptual activitations in the student's reasoning that most of the older students did not demonstrate.

In the second item, students were asked to make a table of values for the graph of an increasing linear function with a negative y-intercept (see Figure 2). The integrated conceptual understanding required here is in students' ability to use numeric and spatial *understandings* and *doings* to show that both the graph and the table for an

Make a table of values that would produce the function seen below.

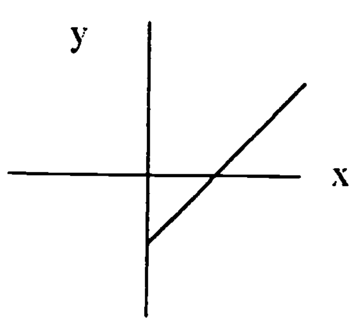


Figure 2. Item 2.

increasing linear function have a constant slope. This is done by generating a table that has a constant increase in y for every unit change in x. It was generally expected that students would produce a correct response by estimating the value of the y-intercept and constructing a table of covarying quantities that increase from there in a linear fashion (e.g., Confrey & Smith, 1995).

Only 18% of the eleventh-grade students gave correct solutions to this item compared to 54% of the sixth graders. There were two common errors for the older students. The first was to estimate coordinate points that would be on the line and simply record those points in a table. This strategy was used without regard for the idea that the x and y values must covary in a certain way -- the y-values must increase at a constant rate for every unit change in x. The second common error resulted from students' difficulty with identifying coordinate pairs on a line. Many students (29% of them) still erroneously determined coordinates by taking the y-value from an estimated y-intercept and the x-value from an estimated x-intercept and calling that a coordinate. The Grade 11 students' reliance on a numeric approach was a tenuous strategy at best, even when an algorithm was known or mastered. When *doings* were flawed, however, such reliance was a major impediment.

The sixth graders, on the other hand, showed a clear understanding that the table needed to have a constant increasing linear pattern, which they manifest with constant covariation between x and y. The limiting factor for these younger students was their difficulty with proficiently computing with negative numbers.

This second task was especially revealing with respect to showing how the textbook-taught students were not attending simultaneously either to the spatial and numeric aspects of a function or to the intrinsic *understandings* and *doings* required for the problem. Rather, they were relying primarily on numeric strategies and with an emphasis on *doing*. On the other hand, the younger students did show an integrated approach to this problem, which suggested again how numeric and spatial schemas and how *understanding* and *doing* are inter-active in sophisticated mathematical reasoning.

Summary and Implications

We used Case's theory of intellectual development and related empirical work on the teaching and learning of functions as a guiding framework to show how conceptual and procedural knowledge relate to each other as children construct an integrated conceptual cognitive structure for understanding in the domain. We argued that *understandings* and *doings* (procedural and conceptual knowledge) are present in some proportion when students are reasoning about sophisticated mathematical ideas such as those found in functions. We also presented the case for how numeric and spatial features of functions must be co-active when creating a central conceptual structure for understanding in a domain such as functions, which includes multiple ways of representing a common concept. Such co-activation may be promoted and facilitated through appropriate curricular and instructional design such as the "walkathon" curriculum presented here, with its meaningful bridge to students' previous *understandings* and *doings*.

The two tasks discussed above are particularly interesting because they show how young students who learned with the experimental curriculum were reasoning in an integrated conceptual fashion using both *understandings* and *doings* and were much more successful than older, more experienced students. The older textbook learners, on the other hand, demonstrated a mostly numeric and *doings* approach to the problems and struggled with these problems.

The implications for this work are many, especially with respect to practice, including curriculum design and classroom teaching. For example, sixth graders were found to be capable of integrating their primary understandings to form an integrated conceptual structure for functions. Thus, meaningful instruction on functions might begin earlier in students' school learning experiences. We would suggest that particular attention be paid to designing and implementing the sort of curricula and instruction used here, which are based in a cognitive theory that promotes the enrichment and integration both of the spatial and numeric aspects of function and of relevant *understandings* and *doings* in order to construct a deep integrated conceptual understanding in the domain. Such an integrated conceptual foundation might help prevent the difficulties found at present among older students' learning of functions.

We have longer-term goals in this endeavor. First, we must try to develop language that will adequately convey the complexities of these issues. We seek to bridge and eventually to blend the procedural-conceptual divide because we find these to be continually and inextricably intertwined. Furthermore, this divide stimulates political debates about learning goals and about pedagogy that, in our view, do not advance the public interest in all children learning mathematics in comprehensible ways. Second, in order to investigate and potentially measure the impact of procedural automaticity on conceptual gains and not-yet-conceptual structures for understanding functions (and other mathematical domains), further cross-sectional and longitudinal work needs to be carried out.

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Note

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