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# SUBTRACTING BY COUNTING UP: MORE EVIDENCE

KAREN C. FUSON, *Northwestern University*
GORDON B. WILLIS, *Northwestern University*

To find any subtraction facts that children did not know (e.g., 14 − 8), 10 teachers successfully taught first- and second-grade children of all ability levels to count up from the smaller to the larger number while keeping track with one-handed finger patterns. Second graders at and above grade level and first graders above grade level in mathematics counted up with finger patterns to solve large multidigit subtraction problems that required regrouping. Learning to subtract by counting up greatly improved children's performance on take-away, compare, and equalize word problems, did not interfere with their understanding of take-away problems, and enabled them to accelerate by as much as 3 years their learning of subtraction topics. Other evidence indicated that subtracting by counting up is natural for children if they learn that one meaning of subtraction (and of the − symbol) is a compare/equalize meaning.

In an earlier paper (Fuson, 1986b), one of us reported positive effects of a conceptually based instructional unit teaching first and second graders to find symbolic subtraction combinations such as 14 − 8 by counting up from the smaller to the larger number using one-handed finger patterns to keep track of the number of words counted up; for example, for 14 − 8, the student says "Eight, nine, ten, eleven, twelve, thirteen, fourteen" while making finger patterns for 1 through 6 for the words said after 8. A number of issues remained from this work. The present study was designed to address several of these issues.

First, the earlier sample had contained as many first graders who were above average as were either average or below average in mathematics, and the results were pooled for analysis across the classes. It seemed important to ascertain the extent to which average and below-average first graders could learn to subtract by counting up.

Second, it seemed possible that learning to count up with finger patterns might have short-term benefits but long-term costs. Many teachers argue that children who are still using their fingers to find differences within 18 cannot learn the multidigit algorithm. This claim in fact may be true for the usual finger methods. They involve fingers on both hands, and children often put

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down their pencil to use these methods (Fuson & Secada, 1986). However, because counting up with finger patterns is done with the fingers on the nonwriting hand, it might be procedurally efficient enough to be used in the multidigit subtraction algorithm. A secondary purpose of the study was to ascertain whether subtracting by counting up would permit children to learn the multidigit algorithm much earlier than is usual in American schools. A recent textbook analysis indicated that American children both begin and complete their study of this topic considerably later than do children in some other countries with strong mathematics education programs: the Soviet Union, Japan, and Taiwan (Fuson, Stigler, & Bartsch, 1988).

Third, there was some concern that learning to subtract by counting up might interfere with children's understanding of take-away subtraction situations. Initially, children solve take-away (separate) problems by modeling with objects the action change structure in such problems: They make a set of entities for the initial large set, take away entities for the change set (separate them from the others), and then count the entities remaining for the answer (Carpenter & Moser, 1983, 1984; Riley, Greeno, & Heller, 1983). Children then usually move on to using counting down to solve take-away problems. In our previous study, children taught to count up solved take-away problems primarily by counting up. In spite of the fact that the initial counting-up instruction showed how to interpret take-away problems as counting-up instead of counting-down situations (see Table 1 in Fuson, 1986b), it seemed possible that these children were applying the counting-up procedure in a rote fashion and were not really understanding the take-away situations.

Fourth, counting up is not an efficient strategy when one is subtracting only 1 or 2 from a much larger number (e.g., $8 - 1$). Although in general counting down is much more difficult than counting up (Baroody, 1984; Fuson, 1984; Fuson, Richards, & Briars, 1982), giving the number that comes immediately before or just two before a number is fairly easy for primary school children (Fuson et al., 1982). Therefore, it seemed sensible to ascertain whether children who are taught to count up will solve problems of the form $x - 1$ and $x - 2$ by thinking of numbers coming before rather than by counting up.

Finally, we wanted to gather more data on relationships between learning to add by counting on and learning to subtract by counting up. In the first study there was evidence that learning to count up led some children to forget how to count on. The present study examined such interference more closely and explored whether some instruction directed at reducing this interference would be effective. The effects of counting-on instruction on subtraction performance of various kinds were also assessed.

In summary, the eight research questions for this study were as follows:

1. Can first graders of average and below-average ability learn to subtract all the most difficult subtraction facts by counting up with one-handed finger patterns?

2. Is counting up with one-handed finger patterns efficient enough to be used in multidigit subtraction and can the latter then be taught earlier than at present?

3. Does learning to solve subtraction facts by counting up interfere with children's understanding of take-away problems?

4. Does learning to solve subtraction facts by counting up decrease children's speed and accuracy on problems of the form $x - 1$ and $x - 2$?

5. Does learning to subtract by counting up interfere with adding by counting on?

6. Can a moderate amount of instruction alleviate such interference?

7. What, if any, general effect does teaching children to add by counting on have on subtraction performance?

8. Do any children invent counting up in a subtraction context after they have learned to count on but before they have been taught to count up?

## METHOD

### *Subjects*

Because the research questions ranged from issues important early in subtraction instruction to those much later in instruction (the multidigit algorithm), the subjects for different questions varied. The subjects for Question 1 were all the first-grade children and all the below-average and average second-grade children in two schools. The school populations were racially and socioeconomically heterogeneous. The children in each grade in each school had been divided by the teachers into three mathematics classes (one above average, one average, and one below average). Thus, there were six classes of first graders and four classes of second graders. The class sizes ranged from 15 to 18 for the below-average classes and from 22 to 27 for the average and above-average classes. The first graders had not participated in the previous study (Fuson, 1986b), but some of the second graders in each school had been subjects in this study and had learned to count up with finger patterns in the first grade. Of the 10 teachers, 5 were familiar with the units on counting up with finger patterns because they had been teachers for the earlier study.

The subjects for Question 2 were the children in the two average second-grade classes, one of the above-average second-grade classes, and one of the above-average first-grade classes. These were the four classes in which the multidigit subtraction instruction was completed or at least partially completed by the end of the school year.

The remaining questions concerned issues important in initial instruction of counting up, so the subjects were drawn from the first-grade classes. Some of these questions required observations of children's behavior, so individual interviews were held with selected children. For Questions 3 to 5, eight children were randomly selected from each of the six first-grade classes. The children were interviewed after the counting-on addition instruction but before the counting-up subtraction instruction and again after the counting-up subtraction instruction. Three children were unavailable for the interview after the counting-up instruction, so the sample was reduced from 48 to 45 for that interview.

The sample for Question 6 consisted of half of the first-grade classes (one class at each ability level).

Questions 7 and 8 required a comparison between children taught to count on and a control group of children who had received no teaching concerning counting on with finger patterns. Two schools in the same district as the experimental schools were identified as having profiles of enrollment and achievement similar to those of the experimental schools. Eight first-grade children above average in mathematics and eight children average in mathematics were randomly selected from one school; five above-average and five average children were randomly selected from the other school because it was smaller than the other schools. We had intended to include below-average children in the control sample, but the teachers of both below-average first-grade control classes strongly opposed the interviews for their students because they said that the interview tasks were quite beyond the capabilities of the children (the tasks all involved minuends above 10, and both teachers taught only minuends to 10). We decided not to pursue these interviews over such opposition. It seemed likely that any effect of the counting-on instruction general enough to affect subtraction performance would appear in the average and above-average children.

### *Instruction, Written Tests, and Interview Tasks*

*Question 1.* The unit on subtraction by counting up was preceded by a unit on counting on with finger patterns for addition of sums between 10 and 18 (see Fuson & Secada, 1986, for details of this teaching and of prerequisites for this unit). The subtraction unit immediately followed the addition unit in all classes except two first-grade classes in which the teachers first spent considerable time on subtraction with minuends below 11 (the other first-grade classes had done this topic earlier in the fall before the counting-on unit). The teachers chose when to present both the counting-on and the counting-up units. The timing of the teaching units and the amount of related instruction that had occurred previously varied considerably among the classes, mainly as a function of the ability level of the class. The counting-on and counting-up units were begun between November and March in the first grade and at the beginning of the year in the second grade.

Minor modifications were made in the counting-up lesson plans and worksheets from the earlier study. Two of the 20 problems on each worksheet were changed to have very small numbers (e.g., $3 - 1$) in order to reinforce the point that finger patterns are used only for problems whose answers one does not already know; the remaining problems had minuends between 10 and 18, and both the subtrahend and the difference were single-digit numbers. The lesson plans were not divided into separate days, so that teachers could move at a pace comfortable for their own class.

The important features of the instruction were that the initial introduction of subtraction was done within the context of the first three kinds of subtraction situations in Table 1, the − symbol was always to be referred to as "minus" (not as "take away"), the children practiced counting up using worksheets containing the most difficult single-digit combinations (except for the above-noted very small numbers), the children practiced problems in both row (horizontal) and column (vertical) form, and there were 10 word problems (8 subtraction and 2 addition problems like the first four problems in Table 1) included at the end of the unit (see Fuson, 1986a, for further details).

Table 1
*Types of Word Problems Used in the Interviews*

| |
|---|
| Take away (missing result) |
| The balloon man had 15 balloons. He gave 7 of the balloons to the children. How many balloons does he have now? |
| Compare (missing difference) |
| The poodle has 12 puppies. The collie has 5 puppies. How many more puppies does the poodle have? |
| Equalize (missing difference) |
| The circus has 14 tigers. The zoo has 8 tigers. How many tigers does the zoo have to get to have as many tigers as the circus has? |
| Change-join (missing result) |
| Matt has 13 dimes. Ann gave him 7 more dimes. How many dimes does Matt have now? |
| Change-join (missing change) |
| The yard had 6 robins in it. Some more robins flew in. Now there are 14 robins in the yard. How many robins flew into the yard? |
| Compare (missing big) |
| Joan had 13 cookies. Susan had 6 more cookies than Joan had. How many cookies did Susan have? |
| Division |
| There were 3 kids, and their mom gave them 12 cookies. If the kids share the cookies equally, how many cookies does each kid have? |

The test of counting up with finger patterns was a timed (2 minute) test of 20 of the most difficult subtraction combinations: The minuends ranged from 11 through 18, both subtrahends and differences ranged from 2 through 9, and no doubles ($a + a$) were included. This test with facts in column (vertical) form was given as a pretest after the counting-on instruction but before the counting-up instruction. Two 2-minute immediate posttests on subtraction were given in counterbalanced order at the completion of the subtraction counting-up teaching. One consisted of 20 facts in row form; the other, 20 facts in column form. These tests were also given 1 and 2 months after instruction to assess whether learning was maintained over these periods. At these administrations, the efficacy of a short teacher review of counting up with finger patterns in remediating any forgetting or slowing of performance was also explored. Therefore the two tests (row and column forms) were given in counterbalanced order at the beginning of a period, the teacher reviewed counting up with finger patterns for 5–10 minutes, and the tests were given again.

*Question 2*. The children in one above-average first-grade class, two average second-grade classes, and one above-average second-grade class were taught a unit on concepts of place value and multidigit addition and a unit on multidigit subtraction. A physical embodiment of the first four base-10 places was used to give meaning to the multidigit addition and subtraction algorithms, and children generalized the algorithm from 4 places to as many as 10 places after they had begun working symbolic problems without the physical embodiment (see Fuson, 1986a, for more details of the instruction and testing). The children were told to use counting on with finger patterns for any addition facts they did not know and to use counting up with finger patterns for any subtraction facts they did not know. The subtraction instruction began between February and May.

The children who were taught the multidigit subtraction algorithm were given various tests. However, only one test was given to all four classes. This test consisted of a single problem written in vertical form:

$$\begin{array}{r} 14943307654 \\ -\ \underline{6385720918} \end{array}$$

This problem was given to test generalization of the multidigit procedure to many places. The problem has three columns in which no trade is necessary so as to test whether children are trading (regrouping, borrowing) only when necessary. One point was given for each correct number in the answer, for a possible score of 10. The two average second-grade classes also were given a four-item test of two- and three-digit problems, a four-item test of four- and six-digit problems, and a six-item test of two-, three-, and four-digit problems with from one to three zeros in the minuend. Each column in which the digit in the answer was correct was given one point, for possible scores on these tests of 10, 21, and 18. The above-average second-grade class did not take these tests because they were not written until after the initial posttesting of that class. The above-average first-grade class did not take them because there was no time left in the school year.

*Question 3*. To assess effects of counting-up instruction on children's understanding of take-away problems, the children were asked during the subtraction preteaching and postteaching interview to solve a compare, an equalize, and a take-away subtraction word problem (see Table 1). The children first solved each problem without any objects available and then were asked to use blocks to solve it. Six different orders of the three problems were used such that each problem was in each position equally often; the same problem order was used for a given child for the problems solved with and without blocks. Two sets of subtraction problems were constructed such that the problem wording was similar but the story topic varied. Half of the children received Problem Set 1 first (for solution without blocks) and Problem Set 2 second (for solution with the blocks), and half received the problem sets in the opposite order (first solving Set 2 without the blocks). The number triplets used in the problems were 15, 7, 8; 12, 5, 7; 14, 8, 6; and 13, 7, 6. The number triplets were crossed with subtraction story type so that over all children all types of problems presented all number triplets. For a given child, the number triplet for a given story type was the same in the stories given without and with blocks. All counterbalancing was crossed with the ability level of the children.

The children were then shown an index card with 13 − 6 written on it. They were asked to find the answer and then to tell a story about 13 − 6.

Paper and a pencil were available to the child throughout the interview. The children were told at the beginning that the interviewer was interested in determining how children solved mathematics problems and that they were free to solve problems in any way they wished.

*Question 4.* After all other tasks in the interview, the children were shown in counterbalanced order cards containing 7 − 1 and 9 − 2 and were asked to give the answer for each. The interviewer recorded the solution strategy the child used.

*Questions 5 and 6.* The tests of symbolic addition performance used to ascertain whether subtraction counting-up instruction interfered with addition performance were identical to the counting-up tests except that the addition test consisted of the 20 addition facts formed from the 20 subtraction facts. These tests were given before and just after the counting-on instruction and at monthly intervals in the test-review-retest procedure described above for subtraction.

A change-join (missing result) addition problem (see Table 1) was given in the subtraction preteaching and postteaching interviews to assess interference from the counting-up instruction. This problem was given between the three subtraction problems without blocks and the three subtraction problems with blocks. Only numbers from the triplets 12, 5, 7 and 13, 7, 6 (from the four triplets given above for the subtraction problems in Question 3) were used for this problem so that the answer would not be greater than 20. The use of similar number triplets for the addition and subtraction problems put the addition problem at a possible disadvantage because the children had to add a single-digit to a two-digit number for addition. Children who counted on for addition could solve the problem fairly readily *if* they started the counting on with the larger number. We felt that the procedure was preferable to using two single-digit numbers in the addition problem and thereby possibly cuing some children to add by the size of the numbers.

The instruction aimed at reducing the interference between counting on and counting up was focused both on differences between these two procedures and on the use of the procedures in simple word-problem situations. It was given after both the counting-on and counting-up instructional units had been completed and after 2 months of test-review-tests had been given for each procedure. Our research associate gave the instruction in one below-average and one average first-grade class near the end of the year. Half a mathematics period each day was used to teach a 2-week unit focused on the simple addition and subtraction problems (the first four kinds of problems in Table 1 and combine [missing whole] addition problems). The children were taught to make a schematic drawing for each kind of word problem and to fill in the known information in the drawing (see Willis & Fuson, 1988, for more details of the teaching method). The instruction for the above-average class was given by the teacher, who briefly reviewed counting on, counting up, and the basic addition and subtraction word problems.

The efficacy of this instruction was tested by interviewing after the instruction the 23 instructed children who were in the original interview sample. The first tasks were the four word problems given in the original interviews. Three other word problems were given to clarify a methodological issue related to the interpretation of the word-problem data from the regular interviews. It had been suggested to us that some children may not have understood that all the subtraction word problems were subtraction problems. Instead they may only have understood that these problems are not addition problems and then may have used the only nonaddition solution procedure they knew (subtraction by counting up). The three new word problems given in the third interview were a division problem, a change-join (missing change) problem, and a compare (missing big) problem (see Table 1). These problems were given in counterbalanced order after the first four problems. No objects were provided for solving any of these problems, though paper and a pencil (and of course fingers) were available throughout the interview.

*Questions 7 and 8.* Children in the control group interviews were given (without blocks) the four word problems given in the interviews for Question 3. These problems were followed by the fact 13 − 6 written on a card.

## RESULTS

### *Question 1: Counting-Up Performance of First-Grade Classes*

The counting-up instruction for subtraction varied in duration across the 10 classes, requiring from 8 to 15 class periods. There was no significant difference in any class, at any test time, between the scores on the column version of the counting-up test and the scores on the row version of that test. For simplicity, therefore, only the results for the column tests will be reported. The mean scores for the pretest, the immediate posttest, and the 1-month and 2-month posttests before and after reviews are given by class in Table 2. The scores in all classes were significantly and very considerably higher on the posttest than on the pretest. In most classes subtraction performance was as accurate and as rapid as addition performance.

Table 2
*Class Means on Symbolic Addition and Subtraction Column Tests*

| | | | | | | 1 Month | | | 2 Month | | 3 Month | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Grade | Ability | School | Pre | Post | | BR | AR | | BR | AR | BR | AR |
| | | | | | | Symbolic Addition | | | | | | |
| 1 | Low | A | 1 | < 9 | | 9 | 11 | s | 9 | 11 | | |
| 1 | Low | B | 2 | < 10 | s | $6^-$ | < 10 | | 9 | 11 | | |
| 1 | Average | A | 2 | < 10 | | $12^+$ | $12^+$ | s | | | | |
| 1 | Average | B | 2 | < 11 | s | 5 | $6^-$ | | | | $8^-$ | < 13 |
| 1 | High | A | 3 | < 13 | s | 13 | < $16^+$ | | | | 14 | $16^+$ |
| 1 | High | B | 7 | < 13 | s | 11 | < 15 | | | | $16^+$ | < $19^+$ |
| 2 | Low | A | 11 | < 14 | s | 14 | $17^+$ | | | | 16 | $17^+$ |
| 2 | Low | B | 7 | < 14 | s | $11^-$ | $11^-$ | | | | 15 | $17^+$ |
| 2 | Average | A | 11 | < 17 | s | 18 | 18 | | | | 18 | 18 |
| 2 | Average | B | 12 | < 17 | s | 16 | 17 | | | | 16 | 15 |
| | | | | | | Symbolic Subtraction | | | | | | |
| 1 | Low | A | 1 | < 11 | | | | | | | | |
| 1 | Low | B | 2 | < 8 | | 9 | 9 | | 8 | 11 | | |
| 1 | Average | A | 2 | < 9 | | | | | | | | |
| 1 | Average | B | 0 | < 11 | | 9 | 9 | | 10 | < 13 | | |
| 1 | High | A | 1 | < 16 | | 16 | 17 | | 16 | < $18^+$ | | |
| 1 | High | B | 2 | < 14 | | 16 | $17^+$ | | $18^+$ | $18^+$ | | |
| 2 | Low | A | 3 | < 13 | | 12 | 16 | | 12 | < 15 | | |
| 2 | Low | B | 2 | < 15 | | 15 | 16 | | 13 | < 17 | | |
| 2 | Average | A | 10 | < 17 | | 18 | 18 | | 18 | $18^+$ | | |
| 2 | Average | B | 7 | < 15 | | 15 | 16 | | $18^+$ | $18^+$ | | |

*Note.* BR is the test given before the review; AR is the test given after the review; "s" denotes the time of the subtraction counting-up instruction. The missing tests for the top three rows were due to the instruction occurring so late in the year that those testing times occurred in the summer. The 2-month addition tests for the seven bottom rows are missing because the testing occurred during Christmas vacation.

< indicates a $t$-test difference significant at the .01 level.
$^-$ indicates a score significantly lower than the posttest ($p < .01$).
$^+$ indicates a score significantly higher than the posttest ($p < .01$).

Almost all the first-grade children interviewed (42 of the 45) spontaneously and correctly used counting up with finger patterns on at least one kind of subtraction problem in the postteaching interview, showing that the children could and did use counting up to subtract. Furthermore, many of the average and below-average first-grade children were quite proud of learning to subtract such large numbers. All teachers reported that their children were much more enthusiastic about subtraction than in the years before the counting-up unit; subtracting by counting up seemed much easier for the children than approaches used in the past.

## *Question 2: Multidigit Subtraction*

Most of the children learning the multidigit subtraction algorithm did not know all their subtraction facts and counted up with finger patterns to find these unknown single-digit differences. The class means on the multidigit subtraction tests are given in Table 3. These test results indicate that the multidigit subtraction instruction was quite successful in all four of the classes (see Fuson, 1986a, for more details including interview results with those children having low scores on these tests). Furthermore, the teachers reported that their children enjoyed learning to subtract such large numbers.

Table 3
*Class Means on Multidigit Subtraction Tests*

| | | | Test | | | |
|---|---|---|---|---|---|---|
| Grade | Ability | School | 2- and 3-digit | 4- and 6-digit | 10-digit | Zeros in minuend |
| 1 | high | A | — | — | 9.7 | — |
| 2 | average | A | 8.3 | 15.8 | 8.5 | 13.2 |
| 2 | average | B | 9.0 | 18.0 | 8.5 | 15.4 |
| 2 | high | A | — | — | 8.5 | — |
| Maximum score | | | 10 | 21 | 10 | 18 |

*Note.* Dashes indicate tests not administered.

## *Question 3: Effects of Teaching Counting Up on Understanding Take-Away Problems*

The effects of counting-up instruction on take-away problems were compared to the effects on compare and equalize problems, which may fit a counting-up model more readily. A $2 \times 2 \times 3$ (Before and After Instruction $\times$ With and Without Blocks $\times$ Problem Type) analysis of variance with repeated measures for all three factors was carried out on strategy scores created by giving one point for each problem on which a child used a strategy that could lead to a correct answer. The same ANOVA was also carried out on answer scores with points given only for correct answers.

These analyses revealed similar results for correct strategies and for correct answers (see Table 4 for mean percent correct in each cell). There was a significant main effect of instruction, $F(1, 44) = 34.22$ and $57.51$, $p < .001$, and of problem type, $F(2, 88) = 9.44$ and $4.30$, $p < .02$, and a significant interaction between instruction and block availability, $F(1, 44) = 4.15$ and $4.71$, $p < .05$. McNemar's tests for correlated proportions[1] indicated that significantly more children used correct strategies (and found correct answers) for subtraction problems after instruction than before, for every

[1]The main effects and the interaction were based on scores with a scale created by the pooling across variables that was done for the analysis (e.g., the Instruction scale ranged from 0 through 6 and the Problem Type scale ranged from 0 through 4). However, each Instruction by Blocks by Problem Type cell contained only one problem, so scores are either 0 or 1. Therefore McNemar's test for correlated proportions was used to locate the effects more precisely.

problem type both with and without blocks provided (see Table 4). Chi-square scores ranged from 4.00 to 19.01, $p < .05$.

Table 4
*Mean Percent Correct for Solution Strategies and Answers on Subtraction Word Problems*

| | Correct strategy | | | | Correct answer | | |
|---|---|---|---|---|---|---|---|
| Time | Take away | | Compare | Equalize | Take away | Compare | Equalize |
| | | | | Without blocks | | | |
| Before instruction | 58 | > | 38 | 44 | 29 | 18 | 29 |
| | Λ | | Λ | Λ | Λ | Λ | Λ |
| After instruction | 93 | > | 78 | 78 | 69 | 60 | 71 |
| | | | | With blocks | | | |
| Before instruction | 60 | > | 42 | 51 | 42 | 29 | 33 |
| | Λ | | Λ | Λ | Λ | Λ | Λ |
| After instruction | 80 | | 71 | 69 | 71 | 53 | 60 |

*Note.* Inverted vees and two underlined numbers indicate proportions that differ significantly by a McNemar test, $p < .05$.

For the take-away problem without blocks, the instruction resulted in a considerable drop in the number of children counting down, a considerable increase in the number counting up, and little change in the number using a take-away object strategy (see Table 5). The data on the take-away strategies with blocks indicate that this increase in counting up was not at the expense of understanding the take-away problem: Even more children modeled a take-away solution with blocks after instruction than before. The instruction even caused an increase in the number of children using a take-away strategy to solve compare and equalize problems with blocks: After instruction four children used take away for all three kinds of subtraction problems and four others used it for two of the problems, for a total of eight such users, whereas only one child had used take away on compare or equalize problems with blocks before the counting-up instruction. These eight children were mostly above-average and used counting up to solve problems without blocks. This use of a block/take-away strategy therefore may reflect a generalization that these problems are all subtraction problems, and take away with blocks is a good general object solution for subtraction problems.

McNemar's tests indicated that significantly more children used a correct strategy on take-away (a) than on compare problems before instruction with blocks, before instruction without blocks, and after instruction without blocks and (b) than on the equalize problems after instruction without blocks (see Table 4). In all four of these cases, these results were largely due to similar levels of use of sequence strategies on take-away and compare or equalize problems but more uses of object strategies on the take-away problem (see Table 5).

Although the overall effect of problem type was significant for correct answers, the McNemar tests on the correct answers were not significant for any comparison (i.e.,the proportion of correct answers did not differ signifi-

Table 5
*Frequency of Correct Strategies Used to Solve Take-Away, Compare, and Equalize Subtraction Word Problems*

| | Before instruction | | | After instruction | | |
|---|---|---|---|---|---|---|
| Strategy | Take away | Compare | Equalize | Take away | Compare | Equalize |
| | | | Without blocks | | | |
| Object | | | | | | |
| Take away | 7 | 0 | 0 | 8 | 2 | 0 |
| Compare | 0 | 1 | 0 | 0 | 2 | 1 |
| Separate sum into two parts | 0 | 1 | 1 | 0 | 0 | 0 |
| Sequence | | | | | | |
| Count down (FPCD) | 15(6) | 1 | 1 | 4(2) | 1 | 0 |
| Count up (FPCU) | 5(3) | 14(11) | 17(13) | 26(25) | 28(25) | 30(26) |
| Known or derived fact | 2 | 1 | 1 | 4 | 3 | 7 |
| | | | With blocks | | | |
| Object | | | | | | |
| Take away | 18 | 1 | 0 | 24 | 6 | 6 |
| Compare | 1 | 6 | 2 | 1 | 9 | 10 |
| Separate sum into two parts | 0 | 2 | 2 | 0 | 0 | 0 |
| Sequence | | | | | | |
| Count down | 5 | 1 | 2 | 0 | 0 | 0 |
| Count up | 5 | 10 | 17 | 10 | 18 | 17 |
| Known or derived fact | 0 | 0 | 0 | 2 | 1 | 2 |

*Note.* Children occasionally used more than one strategy; both are included in the table. Object strategies involve modeling take-away (making the sum and taking away the known part) or compare (making the two known sets and counting the difference) problems or separating the sum objects into two parts; sets were made with fingers, marks on paper, or blocks. FPCD and FPCU are finger pattern count down and finger pattern count up. In sequence strategies with blocks, the blocks were used to keep track of the number counted up or down.

cantly between any two problem types; see Table 4). However, this relatively equal distribution of correct answers across problem types did not result from many children solving all three problems correctly and some solving none correctly. Varied combinations of the three kinds of subtraction problems were solved correctly, and most children (40 of the 45) solved at least one subtraction problem correctly. Thus the evidence did not support the view that children were just mindlessly counting up for every problem.

The Instruction × Block Availability interactions for strategy and for answer were due to similar patterns. Performance before instruction showed the usual relationship of block availability: Performance was better with than without blocks, especially for correct answers. After instruction the reverse pattern was true: The number of correct strategies and answers without blocks was greater than or equal to the number with blocks. This reversal was mainly due to the greater use after instruction of known or derived facts without blocks than with blocks (14 versus 5 uses).

Before instruction, both the well-formed (complete) and the partially formed (accurate but incomplete) stories the children told for the symbolic problem 13 − 6 were evenly divided between take away and equalize. This result is somewhat surprising, for some classes had learned the subtraction of small numbers as take away, and the children had not heard equalize stories before the two stories in the interview. After instruction, the total number of well-formed and partially formed stories was again evenly divided between take away and equalize, but a few more children were able to tell complete take-away stories than complete equalize stories. Therefore the counting-up instruction did not seem to have interfered with the children's first interpretation of the subtraction symbol (−), and these interpretations were equally divided between take-away and equalize situations.

### *Question 4: Solution of x − 1 and x − 2 Problems*

The solution procedures used by children giving correct answers to the symbolic problems 7 − 1 and 9 − 2 are given in Table 6. Many more correct answers were given after the counting-up instruction than before: 40 versus 22 for 7 − 1 and 38 versus 20 for 9 − 2. Of the 17 children counting up with finger patterns to solve one or both of these problems after instruction, only 2 had used the faster count-down procedure (or any procedure) before instruction. Thus, counting up with finger patterns did not replace the successful preinstruction strategies. Rather, the teaching provided a solution method for those children who had not been able to solve the problems before.

Table 6
*Frequency of Solution Procedures for Correct Answers to 7 − 1 and 9 − 2 Before and After Count-Up Instruction*

| | 7 − 1 | | 9 − 2 | |
|---|---|---|---|---|
| Procedure | Before | After | Before | After |
| Gave answer immediately | 22 | 28 | 1 | 7 |
| Counted up with finger patterns | 0 | 11 | 0 | 17 |
| Counted down | 0 | 1 | 11 | 11 |
| Take away | 0 | 0 | 4 | 3 |
| Not identified | 0 | 0 | 4 | 0 |

### *Question 5: Interference Between Counting Up and Counting On*

There is some evidence in Table 2 that in some classes for some children, the teaching of counting up for subtraction did seem to interfere with remembering counting on for addition: The addition score following the subtraction teaching was significantly below the original addition posttest score for some classes. All classes recovered this loss, however, so the interference does seem to be remediable. It also is not inevitable, since some classes showed no such interference. The loss in the other direction (addition interfering with subtraction) does not seem to be so considerable. No class mean on a delayed subtraction posttest dropped significantly below the immediate posttest subtraction mean.

For both addition and subtraction in several cells in Table 2, the after-review score was significantly higher than the before-review score. Although practice on the test is inevitably confounded with review in our procedure, the fact that two tests (a column and a row test) were given both before and after the review would seem to argue somewhat against the conclusion that these results stem from practice alone. The test-review-test procedure may be an effective teaching tool in helping children maintain and later increase the number of addition and subtraction problems they can solve rapidly and accurately, and it may help to reduce interference between the two operations.

On the addition word problem given in the interview, five children who had added in the interview before the counting-up instruction subtracted after the instruction; however, five other children who had not added on this problem before the subtraction teaching now added. Six children perseverated in counting up with finger patterns: They counted up with finger patterns on all three subtraction problems without blocks and on the addition problem. However, this behavior did not necessarily reflect only a rote learning of counting up for subtraction without any real understanding of subtraction problems or procedures. Two used the blocks to model one or more subtraction problems, and four others counted up with blocks to keep track, an adaptation of counting up they invented themselves. However, this behavior clearly did indicate the need for some further reviewing of counting on with these children.

### *Question 6: Instruction Differentiating Counting On and Counting Up*

The instruction differentiating the counting-on and counting-up procedures and demonstrating their use in simple word problems was quite successful. The number of correct strategies (and correct answers) for the reinterviewed children is shown in Table 7. After the special instruction, almost all problems were solved by counting on or counting up with finger patterns. This sample contained five of the six subtraction perseveraters; they all added for the addition problem after this instruction.

Table 7
*Frequency of Solution Strategies and Correct Answers on Subtraction Word Problems for Reinterviewed Children (n = 23)*

| | Correct strategy | | | | Correct answer | | | |
|---|---|---|---|---|---|---|---|---|
| Time | Take away | Compare | Equalize | Addition | Take away | Compare | Equalize | Addition |
| After instruction | 23 | 20 | 19 | 12 | 18 | 16 | 16 | 10 |
| Reinterview | 23 | 22 | 21 | 21 | 23 | 21 | 19 | 20 |

Performance on the last three word problems given in the interview after the special interference instruction permitted a clarification of the interpretation of these results. The number of correct strategies and correct answers (of the possible 23) on these problems was 17 and 17 on the change-join (missing change) problem, 16 and 14 on the compare (missing big) problem, and 9 and 8 on the division problem (see Table 1). Five children added for the division problem, and five children subtracted. Across the seven word problems given in this special interview, the response pattern of only one child fit the pattern of adding for the addition problems and subtracting on all others, whether one considers the addition problems as including only the easy problem from the original interview or also as including the compare (missing big) problem. Thus, the excellent performance of these children on the subtraction problems was not due just to a strategy of counting up on any nonaddition problem.

*Question 7: Effects of Counting-On Addition Instruction on Subtraction Strategies*

The strategies used by the control children and the experimental children after the counting-on instruction but before the counting-up instruction were classified as object strategies (direct modeling) or as sequence strategies (counting up or down). Chi-square tests indicated that significantly more experimental than control children used sequence strategies on each problem listed in Table 8, with chi-square values ranging from 4.04 to 16.59, $p < .05$. As reported by earlier investigators (Carpenter & Moser, 1983, 1984; Riley et al., 1983), the strategies used by both groups of children varied with the type of problem. Almost all the solutions to the take-away problem involved counting down or take away, and those to the compare and equalize problems involved counting up or comparing two sets. The object strategies for 13 − 6 were all take away, and most of the 16 sequence solutions were counting down (3 were counting up). Thus, teaching counting on did not seem to change the basic interpretation of subtraction word problems or approaches to symbolic subtraction facts but only the sophistication of the solution strategy. This effect of the addition counting-on instruction—moving the experimental children on to try sequence solution strategies for a range of subtraction situations—was less strong in the below-average experimental children (they were not included in the experimental sample in Table 8 in order to control for the ability level of the control sample). Over all four subtraction problems, the ratio of sequence to object strategies was 18 to 8 for the below-average children compared to 54 to 9 for the average and above-average children appearing in Table 8.

Table 8
*Frequency of Correct Solution Strategies Used by Control and Preteaching Experimental Children*

| Group | *n* | Take-away word problem | | | Compare word problem | | | Equalize word problem | | | 13 − 6 | | | Addition word problem | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Obj | Seq | Fact | Obj | Seq | Fact | Obj | Seq | Fact | Obj | Seq | Fact | Obj | Seq | Fact |
| Control | 26 | 15 | 4 | 4 | 3 | 3 | 2 | 4 | 6 | 2 | 18 | 3 | 1 | 6 | 9 | 4 |
| Experimental | 32 | 4 | 15 | 2 | 1 | 11 | 1 | 1 | 15 | 1 | 3 | 13 | 1 | 2 | 23 | 0 |

*Note.* Object (Obj) strategies are taking away fingers or marks on paper and separating marks on paper or fingers into two sets. Sequence (Seq) strategies are counting down and counting up. Fact strategies are use of derived facts or known facts. Samples consist of first graders of average and above-average mathematical ability.

*Question 8: Invention of Counting Up*

A substantial number of children counted up using finger patterns to keep track before being taught to do so. In the interviews before the counting-up instruction, 18 of the 43 children (42%) counted up using finger patterns at least once for a problem given without blocks. Of these 18, 12 children (67%) counted up correctly. This result is in contrast to the 15 children who counted down in the same interview for a take-away problem; 7 (47%) of these count-down attempts were accurate. Four other children counted up but used fingers, not finger patterns, to keep track. Thus 22 children (51% of the sample) counted up for a subtraction word problem and 68% did so correctly, whereas 15 children (35% of the sample) counted down for a subtraction word problem and only 47% of these did so accurately. Counting up was used much more frequently with the compare and equalize problems than with the take-away problem (see Table 5).

## DISCUSSION AND CONCLUSIONS

To summarize, the answers to the eight research questions are as follows:

1. First-grade children of average and below-average mathematical ability can learn to count up with one-handed finger patterns to solve the most difficult single-digit subtraction facts.

2. Teaching counting up with finger patterns is beneficial to more advanced subtraction calculation: Such counting up is efficient enough to be used by second graders and by above-average first graders to subtract numbers with as many as 10 places. Therefore, children clearly do not have to wait until they have learned their subtraction facts to learn multidigit subtraction.

3. Multiple lines of interview evidence indicate that counting-up instruction does not interfere with children's understanding of take-away situations and even improves it.

4. The counting-up instruction does not change the strategy of those children who before instruction solve facts of the form $x - 1$ and $x - 2$ by giving the number just before or just two before. However, because children who do not solve those problems before instruction tend to count up to solve them after instruction, the more rapid strategies might be discussed with children after counting-up learning is completed.

5. For some classes and some children, there is interference between addition and subtraction, in the understanding of both word problems and the sequence solution procedures of counting on and counting up. It is not clear whether this interference is any worse with counting on and counting up than with other methods of adding and subtracting because no control group was available to assess such interference in "ordinary" subtraction teaching.

6. For average and below-average first graders, the brief word-problem instruction indicated that this interference may be eliminated through specific

teaching that focuses directly on the differences between addition and subtraction word problems and on the differences between the sequence-solution procedures. For the other children, the biweekly alternating addition and subtraction test-review-test sessions appear to have been sufficient to remove any interference and to enable children to reach high levels of speed and accuracy on symbolic addition and subtraction performance.

7. Teaching children to count on with finger patterns for addition seems to move many of them on to a new developmental level, to Fuson's (1988) Level III in which addition and subtraction problems are represented by the number-word sequence rather than by objects. This effect is stronger for children above average and average in mathematics than those below average.

8. Many children can be expected to invent counting up after count-on instruction but before count-up instruction; most of these inventions occur on compare and equalize word problems.

These conclusions indicate that there are many advantages to teaching children to solve symbolic subtraction problems by counting up with one-handed finger patterns. With this approach, children can rapidly and accurately solve the most difficult single-digit subtraction facts by the end of the first grade, considerably earlier than in some textbook series in which the most difficult single-digit subtraction combinations are not even introduced until the second grade (Fuson et al., 1988). Children can learn multidigit subtraction of large numbers in second grade, considerably earlier than is common in this country but much more in line with learning in Asian countries and the Soviet Union (Fuson et al., 1988); this calculating ability is accompanied by an understanding of the multidigit procedure (Fuson, 1986a; Fuson & Briars, 1988). Both first and second graders seem empowered by, and are very enthusiastic about, their ability to solve these "large" problems. First graders show increased ability to solve compare, equalize, and take-away word problems with facts to 18 and are as good at solving these problems without directly modeling them with objects as with such direct modeling. Counting up is so easy that subtraction-fact performance is equivalent to addition-fact performance.

Counting up also seems to be a fairly natural strategy for children in the context of compare and equalize situations because before the counting-up instruction, more children counted up for these problems than counted down for a take-away word problem. It also seems to be quite easy to give an equalize meaning to the subtraction symbol (−) and thus to suggest counting up for symbolic subtraction facts, as indicated by the somewhat surprising result that as many children told an equalize as a take-away story for 13 − 6, even though some of them had already learned a take-away interpretation for −.

The issue in teaching all mathematical symbols is what meaning or meanings to give to these symbols. All too often, school mathematics in operation opts for simplicity at the cost of richness and flexibility. Subtraction (and the − symbol) initially, and in many texts for a long time, is given only one interpretation, that of take away. When children begin inventing sequence strategies, this interpretation leads them to count down to solve symbolic subtraction problems. Even though counting up is considerably easier than counting down, we think that it would be unfortunate if the single compare/equalize interpretation of − is substituted for the take-away interpretation. Instead, symbolic subtraction ought to be first introduced as having several meanings: at least the take-away meaning and the compare/equalize meaning. These meanings can be developed through experience with take-away and compare/equalize situations. Alternative ways to solve given addition or subtraction facts can be discussed by and with children, including object, sequence, and derived-fact solutions that stem from the take-away meaning (e.g., taking away, counting down, down over 10) and from the compare/equalize meaning (e.g., adding on, counting up, doubles plus 1, and up over 10). However, solution procedures other than counting up are difficult for subtraction facts with minuends over 10: It is difficult to show a minuend over 10 with fingers (in take away), counting down is complex, and using derived facts is difficult even for many second graders (Steinberg, 1983/1984, 1985). Thus it seems sensible to ensure that all children have an opportunity to invent or to learn to count up with one-handed finger patterns for these more difficult facts. However, this opportunity ought to be preceded by conceptually based counting-on learning activities that enable children to reflect on their knowledge of counting (see Fuson & Secada, 1986) and ought to be well grounded in meaningful subtraction object situations rather than being taught as a rote procedure for solving symbolic problems.

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AUTHORS

KAREN C. FUSON, Professor, School of Education and Social Policy, Northwestern University, Evanston, IL 60208

GORDON B. WILLIS, Research Associate, School of Education and Social Policy, Northwestern University, Evanston, IL 60208